Exercise 6

Show that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating an appropriate branch of the multiple-valued function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1}$$

over (a) the indented path in Fig. 101, Sec. 82; (b) the closed contour in Fig. 103, Sec. 84.

Solution

Part (a)

In order to evaluate this integral, consider the given function in the complex plane $f(z)$ and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

$$z^2 + 1 = 0$$
$$z = \pm i$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$. Since $z^{-1/2}$ can be written in terms of $\log z$, a branch cut for the function needs to be chosen.

$$z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right)$$

We choose it to be the axis of negative imaginary numbers.

$$= \exp\left[-\frac{1}{2}(\ln r + i\theta)\right], \quad \left(|z| > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z.

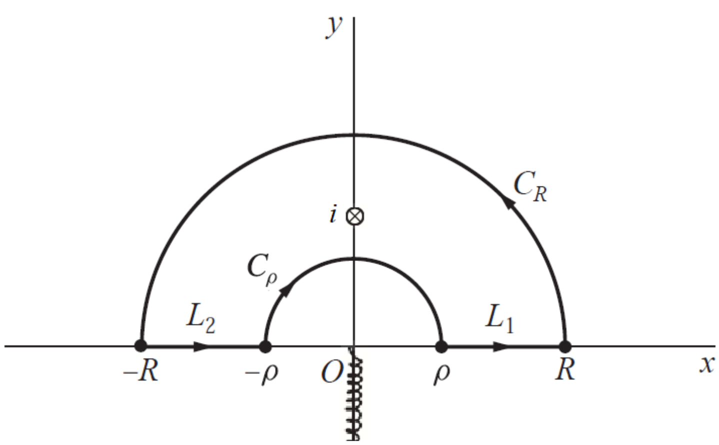

Figure 1: This is Fig. 101 with the singularity at $z = i$ marked. The squiggly line represents the branch cut $(|z| > 0,\ -\pi/2 < \theta < 3\pi/2)$.

According to Cauchy's residue theorem, the integral of $z^{-1/2}/(z^2+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{-1/2}}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1}$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{L_2} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned}
L_1: &\quad z = re^{i0}, && r=\rho \quad \to \quad r=R \\
L_2: &\quad z = re^{i\pi}, && r=R \quad \to \quad r=\rho \\
C_\rho: &\quad z = \rho e^{i\theta}, && \theta=\pi \quad \to \quad \theta=0 \\
C_R: &\quad z = Re^{i\theta}, && \theta=0 \quad \to \quad \theta=\pi
\end{aligned}$$

As a result,

$$\begin{aligned}
2\pi i \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} &= \int_\rho^R \frac{(re^{i0})^{-1/2}}{(re^{i0})^2+1}(dr\,e^{i0}) + \int_R^\rho \frac{(re^{i\pi})^{-1/2}}{(re^{i\pi})^2+1}(dr\,e^{i\pi}) + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz \\
&= \int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_R^\rho \frac{r^{-1/2}e^{-i\pi/2}}{(-r)^2+1}(-dr) + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz \\
&= \int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_\rho^R \frac{r^{-1/2}e^{-i\pi/2}}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz \\
&= (1+e^{-i\pi/2})\int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz.
\end{aligned}$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$(1+e^{-i\pi/2})\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr = 2\pi i \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1}$$

Writing the denominator as $z^2+1 = (z+i)(z-i)$, we see that the multiplicity of the $z-i$ factor is 1. Consequently, the residue at $z=i$ can be calculated by

$$\operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} = \phi(i),$$

where $\phi(z)$ is the same function as $f(z)$ without the $z-i$ factor.

$$\phi(z) = \frac{z^{-1/2}}{z+i} \quad \Rightarrow \quad \phi(i) = \frac{i^{-1/2}}{2i} = \frac{(e^{i\pi/2})^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i}$$

So then

$$\operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} = \frac{e^{-i\pi/4}}{2i}$$

and

$$\begin{aligned}(1+e^{-i\pi/2})\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr &= 2\pi i\left(\frac{e^{-i\pi/4}}{2i}\right)\\ &= \pi e^{-i\pi/4}.\end{aligned}$$

Divide both sides by $1+e^{-i\pi/2}$.

$$\begin{aligned}\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr &= \pi\frac{e^{-i\pi/4}}{1+e^{-i\pi/2}}\\ &= \pi\frac{1}{e^{i\pi/4}+e^{-i\pi/4}}\\ &= \pi\frac{1}{2\cos(\pi/4)}\\ &= \pi\frac{1}{\sqrt{2}}\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}\,dx = \frac{\pi}{\sqrt{2}}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small semicircular arc in Figure 1 is $z = \rho e^{i\theta}$, where θ goes from π to 0.

$$\begin{aligned} \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz &= \int_\pi^0 \frac{(\rho e^{i\theta})^{-1/2}}{(\rho e^{i\theta})^2+1}(\rho i e^{i\theta}\,d\theta) \\ &= \int_\pi^0 \frac{\rho^{1/2}e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta) \end{aligned}$$

Take the limit of both sides as $\rho \to 0$.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = \lim_{\rho\to 0}\int_\pi^0 \frac{\rho^{1/2}e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta)$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = \int_\pi^0 \lim_{\rho\to 0}\frac{\rho^{1/2}e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta)$$

Because of $\rho^{1/2}$ in the numerator, the limit is zero. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large semicircular arc in Figure 1 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned} \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz &= \int_0^\pi \frac{(Re^{i\theta})^{-1/2}}{(Re^{i\theta})^2+1}(Rie^{i\theta}\,d\theta) \\ &= \int_0^\pi \frac{R^{1/2}e^{i\theta/2}}{R^2e^{2i\theta}+1}(i\,d\theta) \\ &= \int_0^\pi \frac{R^{1/2}e^{i\theta/2}}{R^2\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta) \\ &= \int_0^\pi \frac{1}{R^{3/2}}\frac{e^{i\theta/2}}{\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta) \end{aligned}$$

Take the limit of both sides as $R \to \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = \int_0^\pi \lim_{R\to\infty}\frac{1}{R^{3/2}}\frac{e^{i\theta/2}}{\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta)$$

Because of $R^{3/2}$ in the denominator, the limit is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = 0.$$

Part (b)

In order to evaluate this integral, consider the given function in the complex plane $f(z)$ and the contour in Fig. 101. Singularities occur where the denominator is equal to zero.

$$z^2 + 1 = 0$$
$$z = \pm i$$

Since $z^{-1/2}$ can be written in terms of $\log z$, a branch cut for the function needs to be chosen.

$$z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right)$$

We choose it to be the axis of positive real numbers.

$$= \exp\left[-\frac{1}{2}(\ln r + i\theta)\right], \quad (|z| > 0,\ 0 < \theta < 2\pi),$$

where $r = |z|$ is the magnitude of z and $\theta = \arg z$ is the argument of z.

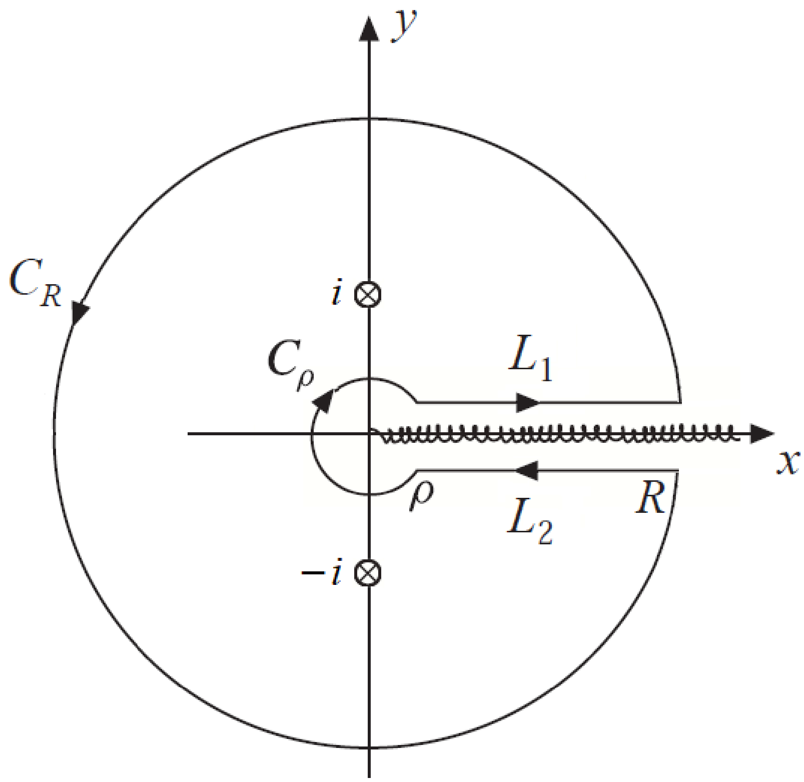

Figure 2: This is essentially Fig. 103 with the singularities at $z = -i$ and $z = i$ marked. The squiggly line represents the branch cut $(|z| > 0,\ 0 < \theta < 2\pi)$.

According to Cauchy's residue theorem, the integral of $z^{-1/2}/(z^2+1)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^{-1/2}}{z^2+1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=-i}\frac{z^{-1/2}}{z^2+1} + \operatorname*{Res}_{z=i}\frac{z^{-1/2}}{z^2+1}\right)$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{L_2} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz$$
$$= 2\pi i \left(\operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} + \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} \right) \quad (1)$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1: &\quad z = re^{i0}, && r=\rho \quad \to \quad r=R \\ L_2: &\quad z = re^{i2\pi}, && r=R \quad \to \quad r=\rho \\ C_\rho: &\quad z = \rho e^{i\theta}, && \theta=2\pi \quad \to \quad \theta=0 \\ C_R: &\quad z = Re^{i\theta}, && \theta=0 \quad \to \quad \theta=2\pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{L_1} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{L_2} \frac{z^{-1/2}}{z^2+1}\,dz &= \int_\rho^R \frac{(re^{i0})^{-1/2}}{(re^{i0})^2+1}\,(dr\,e^{i0}) + \int_R^\rho \frac{(re^{i2\pi})^{-1/2}}{(re^{i2\pi})^2+1}\,(dr\,e^{i2\pi}) \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_R^\rho \frac{r^{-1/2}e^{-i\pi}}{r^2+1}\,dr \\ &= \int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr \\ &= 2\int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr. \end{aligned}$$

Substitute this formula into equation (1).

$$2\int_\rho^R \frac{r^{-1/2}}{r^2+1}\,dr + \int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz + \int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = 2\pi i \left(\operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} + \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} \right)$$

Take the limit now as $\rho \to 0$ and $R \to \infty$. The integral over C_ρ tends to zero, and the integral over C_R tends to zero. Proof for these statements will be given at the end.

$$2\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr = 2\pi i \left(\operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} + \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} \right)$$

Writing the denominator as $z^2+1 = (z+i)(z-i)$, we see that the multiplicities of the $z-i$ and $z+i$ factors are both 1. Consequently, the residues at $z=-i$ and $z=i$ can be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} &= \phi_1(-i) \\ \operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} &= \phi_2(i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are the same function as $f(z)$ without the $z+i$ and $z-i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^{-1/2}}{z-i} \quad \Rightarrow \quad \phi_1(-i) = \frac{(-i)^{-1/2}}{-2i} = \frac{(e^{i3\pi/2})^{-1/2}}{-2i} = -\frac{e^{-i3\pi/4}}{2i} \\ \phi_2(z) &= \frac{z^{-1/2}}{z+i} \quad \Rightarrow \quad \phi_2(i) = \frac{i^{-1/2}}{2i} = \frac{(e^{i\pi/2})^{-1/2}}{2i} = \frac{e^{-i\pi/4}}{2i} \end{aligned}$$

So then

$$\operatorname*{Res}_{z=-i} \frac{z^{-1/2}}{z^2+1} = -\frac{e^{-i3\pi/4}}{2i}$$
$$\operatorname*{Res}_{z=i} \frac{z^{-1/2}}{z^2+1} = \frac{e^{-i\pi/4}}{2i}$$

and

$$\begin{aligned} 2\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr &= 2\pi i\left(-\frac{e^{-i3\pi/4}}{2i} + \frac{e^{-i\pi/4}}{2i}\right) \\ &= \pi(e^{-i\pi/4} - e^{-i3\pi/4}) \\ &= \pi\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} - \left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)\right] \\ &= \pi\left[\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right] \\ &= \pi\sqrt{2}. \end{aligned}$$

Divide both sides by 2.

$$\int_0^\infty \frac{r^{-1/2}}{r^2+1}\,dr = \frac{\pi}{\sqrt{2}}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}\,dx = \frac{\pi}{\sqrt{2}}.}$$

The Integral Over C_ρ

Our aim here is to show that the integral over C_ρ tends to zero in the limit as $\rho \to 0$. The parameterization of the small circular arc in Figure 2 is $z = \rho e^{i\theta}$, where θ goes from 2π to 0.

$$\begin{aligned}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz &= \int_{2\pi}^{0} \frac{(\rho e^{i\theta})^{-1/2}}{(\rho e^{i\theta})^2+1}(\rho i e^{i\theta}\,d\theta) \\ &= \int_{2\pi}^{0} \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta)\end{aligned}$$

Take the limit of both sides as $\rho \to 0$.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = \lim_{\rho\to 0}\int_{2\pi}^{0} \frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta)$$

The limits of integration are constant, so the limit may be brought inside the integral.

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = \int_{2\pi}^{0} \lim_{\rho\to 0}\frac{\rho^{1/2} e^{i\theta/2}}{\rho^2 e^{2i\theta}+1}(i\,d\theta)$$

Because of $\rho^{1/2}$ in the numerator, the limit is zero. Therefore,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{z^{-1/2}}{z^2+1}\,dz = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the large circular arc in Figure 2 is $z = Re^{i\theta}$, where θ goes from 0 to 2π.

$$\begin{aligned}\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz &= \int_{0}^{2\pi} \frac{(Re^{i\theta})^{-1/2}}{(Re^{i\theta})^2+1}(Rie^{i\theta}\,d\theta) \\ &= \int_{0}^{2\pi} \frac{R^{1/2} e^{i\theta/2}}{R^2 e^{2i\theta}+1}(i\,d\theta) \\ &= \int_{0}^{2\pi} \frac{R^{1/2} e^{i\theta/2}}{R^2\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta) \\ &= \int_{0}^{2\pi} \frac{1}{R^{3/2}}\frac{e^{i\theta/2}}{\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta)\end{aligned}$$

Take the limit of both sides as $R \to \infty$. Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = \int_{0}^{2\pi} \lim_{R\to\infty}\frac{1}{R^{3/2}}\frac{e^{i\theta/2}}{\left(e^{2i\theta}+\frac{1}{R^2}\right)}(i\,d\theta)$$

Because of $R^{3/2}$ in the denominator, the limit is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{z^{-1/2}}{z^2+1}\,dz = 0.$$